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Randomly forced Rayleigh-Bénard convection

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We consider the onset of Rayleigh–Bénard convection from random fluctuations arising within a fluid. In the specific case in which the fluctuations are thermodynamically determined, we reduce the problem to a random initial value problem for the Fourier modes. For the case of weak nonlinear convection, it is possible to truncate the number of modes and this truncated set is solved both by a Monte Carlo technique and by moment methods for various Rayleigh numbers. We find three stages in the evolution of ordered convection from random fluctuations which correspond to time intervals in which the fluctuations and the nonlinearity have different degrees of importance. It is shown that no simple moment truncation method will succeed and that the time for onset of convection is a mean over a distribution of times for which members of an ensemble exhibit appreciable convective transport.

1. Introduction

Conventional hydrodynamic stability theory is concerned with the determination of critical values of parameters, such as the Rayleigh number or the Taylor number, demarking a region of stability from that of instability. For supercritical parameter values, the stationary solution becomes unstable to a different finite amplitude solution, which in some cases is sufficiently simple that it can be determined employing well-developed techniques of weak nonlinear stability theory. It is the subject of dynamic instability theory to study the manner in which the new finite amplitude solution evolves from ubiquitous disturbances in the fluid. Questions in the domain of such a transient instability theory include that of non-uniqueness of the new state, that is to say, which of the possible new states will be observed? Other questions include that of a proper description of the transition time for evolution between states and the time at which disturbances become manifest, denoted as the onset time. In its present stage of development, dynamic instability theory is incapable of providing satisfactory answers to some of these questions.

The present work couches these questions in the context of a full statistical description of the random nature of disturbances, which play an important role during the evolution of a transient system to a new stable state. Such statistical characterization of disturbances leads to new concepts in stability theory. By modelling random perturbations in the dynamic system as Gaussian white noise, Ludwig (1975) has shown that the deterministic concept of stability no longer applies. Even if the corresponding deterministic system has an asymptotically stable equilibrium point, there may be a finite probability that random effects will move the system out of the domain of attraction of the equilibrium point. In such a case, the system will eventually

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leave the domain of attraction with probability one. The deterministic concept of stability is then replaced by the expected time elapsed before leaving the domain of attraction. Ludwig refers to this time as the persistence of the system.

The study of random convection problems is in its infancy, although there is ample theoretical and experimental evidence to motivate such studies. For example, Clever & Busse (1978) and Busse & Clever (1979) have demonstrated that simple concepts of the relative stability of one finite amplitude state with respect to another is insufficient to resolve the question of non-uniqueness. This is reflected for example, in the differences between the convection patterns observed in steady experiments when the initial disturbances are controlled (and therefore characterized) and when they are uncontrolled and presumably random (Busse & Whitehead 1974). Recent work in the study of convection in porous materials by Horne (1979) and Straus & Schubert (1979) has also indicated the existence of multiple steady solutions at a given Rayleigh number, with each solution stable relative to the others.

There is even greater motivation for the study and characterization of random disturbances in the study of transitional flows which are linearly unstable, and in the stability of time-dependent flows. In the case of base states which vary periodically in time, theory and experiment are in wide disagreement in some cases; see Davis (1976) for a recent review. It has been suggested on the basis of experimental data, that the presence of random noise may drive a system continuously away from its base state, even though conventional deterministic models would predict some degree of stability; see Finucane & Kelly (1976). The unsatisfactory status of theories of onset time for impulsively driven flows is unchanged since it was critically reviewed by Homsy (1973). Trajectories based upon a deterministic view show a marked sensitivity to initial data, which leads to an indeterminancy in the onset time; see for example Gresho & Sani (1971). More recently, in a study of convection in a rotating layer heated from below, Busse & Clever (1978) have suggested that the inclusion of random noise is necessary for the correct prediction of states of convection characterized by their relative roll orientations, all of which are continuously unstable to each other. Thus a consideration of the random nature of fluctuations has potential impact on many problems in stability theory.

There has been recent interest in the theory of random convection problems. Newell, Lange & Aucoin (1970) treated the problem of the evolution of a random initial spectrum or bandwidth of modes. Assuming the initial distribution to be sufficiently close to Gaussian for cumulant truncation methods to apply, they were able to show that the roll structure which ultimately evolves is one of perfect order. Zaitsev & Shliomis (1971) showed that the usual critical point, R_c, of linear instability theory is unaffected by the addition of random forcing, but that the forcing does lead to an imperfect bifurcation near the eigenvalue of linear theory. Later, Graham & Pleiner (1975) attempted to resolve the mathematical difficulties which occur in the randomly-forced problem in an extremely small region near R_c. Graham (1974) has shown that in a small neighbourhood of the critical point steady solutions to the randomly-forced problem may be represented by a generalized potential, and that this potential is extremized for rolls of perfect order with wavenumber equal to the critical wavenumber. These results thus complement those of Newell *et al.* (1970). Finally it is worth mentioning the intriguing and presently unexplained observations by Ahlers & Behringer (1978) that continuous, aperiodic and measurable fluctuations

occur in a convection apparatus of large aspect ratio as soon as the critical point is exceeded. There is thus ample motivation for the development of theoretical methods for solving random convection problems.

It is the goal of the present work to develop such methods applied to the evolution of cellular convection. In an experimental system, statistically correlated disturbances may be present owing to vibration of the mechanical equipment attached to the system or due to imperfect control of the boundary data. In the absence of all disturbances of mechanical nature, thermodynamic fluctuations become responsible for triggering any instability. Since thermodynamic fluctuations are always present in any macroscopic system, the time evolution of the convection provides an upper bound to the transition time for the establishment of steady convection. In order to demonstrate the analysis with as much mathematical simplicity as possible, we consider the following transient experiment. Let the initial state of the fluid be motionless with an adversely linear temperature profile and a hydrostatic pressure gradient so that the Rayleigh number of the system exceeds its critical value. (Although this system is potentially unstable and thus far from equilibrium, in order to describe the initial disturbances as equilibrium thermodynamic fluctuations we consider the initial state to be in pseudo-thermal equilibrium at some average temperature $\bar{T}$.) Then any infinitesimally small disturbances present in the system amplify as the system evolves to a new stable solution. For weakly supercritical conditions this new solution consists of two-dimensional roll cells which are periodic in the horizontal direction (Schlüter, Lortz & Busse 1965). In this paper we study the transient evolution of the system to these roll cells and evaluate the onset time for convection. In §2 we develop the random-amplitude evolution equations, and specify the statistics of both the random forcing and the initial conditions. Section 3 contains certain results of linear theory which have a later use. In §4 we separate the terms of different order for weakly supercritical conditions. Finite-amplitude steady solutions to lowest order are obtained from simultaneous consideration of the Fockker–Planck equations and the moment equations. With neglect of superimposed thermodynamic fluctuations, this steady state solution is identical to that of Malkus & Veronis (1958) to lowest order, with a probabilistic description of flow direction. Two different methods of obtaining the time dependent solution are outlined in § 5, namely a moment truncation method and a Monte Carlo simulation. It is shown that no simple moment truncation is uniformly valid in time. Three regimes of evolution are identified and discussed.

2. Mathematical formulation

(*a*) *Governing equations*

As we have stated, our interest is in convective states of motion which evolve from random forcing within the fluid. Much of the development which follows is valid for any fluctuations whose statistics are described by (2.2) below: however, in order to fix ideas, we follow several recent researchers in taking the fluctuations to be thermodynamic in nature. Restricting the fluctuations on this steady state to be of thermodynamic origin introduces thermodynamically determined fluctuating forces into the governing equations. These equations were introduced and analysed by Landau & Lifshitz (1959) and further discussed by Keizer (1978) and Fox & Uhlenbeck (1970).

Thus the dimensionless equations for the fluctuating quantities are

$$\left.\begin{aligned} \frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j} &= \frac{-\partial p}{\partial x_i}+PrT\delta_{i3}+Pr\frac{\partial^2 u_i}{\partial x_j\partial x_j}+\frac{\partial S_{ij}}{\partial x_j},\\ \frac{\partial T}{\partial t}+u_j\frac{\partial T}{\partial x_j} &= Ru_i\delta_{i3}+\frac{\partial^2 T}{\partial x_j\partial x_j}-\frac{\partial q_i}{\partial x_i},\\ \frac{\partial u_j}{\partial x_j} &= 0, \end{aligned}\right\} \tag{2.1}$$

where $R = g\beta\Delta Tl^3/\nu\kappa$ and $Pr = \nu/\kappa$, are, respectively the Rayleigh number and the Prandtl number. Here S_{ij} and q_i are the random stress tensor and random heat-flux vector respectively. Correlations of the dimensionless random quantities are given as,

$$\left.\begin{aligned} &\langle S_{ij}(x_p,t_1)\,S_{lm}(x_q,t_2)\rangle = 2\theta\delta(x_p-x_q)\,\delta(t_1-t_2)\,(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl}),\\ &\langle q_i(x_p,t_1)\,q_j(x_q,t_2)\rangle = 2\phi\delta(x_p-x_q)\,\delta(t_1-t_2)\,\delta_{ij},\\ &\langle S_{ij}\,q_l\rangle = \langle S_{ij}\rangle = \langle q_l\rangle = 0, \end{aligned}\right\} \tag{2.2}$$

where $\theta = Pr^3 k\bar{T}/\rho\nu^2 l$ and $\phi = [k\bar{T}^2 g\beta/\rho c_p\nu^2\Delta T]\,(Pr\,R)$. As is well known, $\theta \gg \phi$ for most fluids, and so we neglect q_i in what follows. Neglecting the fluctuations at the boundary, the boundary conditions at $x_3 = 0, 1$ are,

$$u_3 = T = \partial u_1/\partial x_3 = \partial u_2/\partial x_3 = 0. \tag{2.3}$$

(*b*) *Fourier representation*

For a supercritical Rayleigh number R, as is well established by nonlinear stability theory (Schlüter *et al.* 1965), convection sets in in a form of two-dimensional rolls with period $2\pi/\alpha$ in horizontal direction. Since we want to describe the time evolution of this periodic solution from thermodynamic fluctuations to ordered convection, we restrict the analysis to two dimensions and represent dependent random variables as a double sum of Fourier components with period $2\pi/\alpha$ in the x direction, 2 in the z direction and random time-dependent Fourier amplitudes. For other boundary conditions, expansion in terms of eigenfunctions of the stationary Rayleigh–Bénard problem is possible: Zaitsev & Shliomis (1971). For free–free conditions, these eigenmodes are the harmonic functions $\exp(in\pi z)$. Changing the notation from (x_1, x_2, x_3) to (x, y, z) and (u_1, u_2, u_3) to (u, v, w), restricting to two dimensions (x, z), and taking into consideration the continuity equation and the boundary conditions, the Fourier expansions of dependent variables take the forms

$$w = \sum_{a=1}^{\infty}\sum_{n=1}^{\infty} c_{an}(t)\cos(a\alpha x)\sin(n\pi z), \tag{2.4}$$

$$T = \sum_{a=0}^{\infty}\sum_{n=1}^{\infty} d_{an}(t)\cos(a\alpha x)\sin(n\pi z), \tag{2.5}$$

with associated expansions for u and p. It may be noted that the temperature T defined by (2.5) includes a mean field independent of x. This represents the modification of the mean temperature distribution by convection. On the other hand, no mean wind is produced by the convection; therefore $a \neq 0$ in (2.4).

Substituting (2.4), (2.5) into (2.1), using the orthogonality property of the spatial

functions in the fundamental region ($0 \leqslant x \leqslant 2\pi/\alpha$, $0 \leqslant z \leqslant 1$) and eliminating the pressure from the x and the z components of the momentum equations, we obtain, in the usual way, the equations for the time evolution of the random Fourier amplitudes, viz.

$$\begin{aligned}\frac{d}{dt}(c_{cp}(t)) = &\sum_{\substack{a,n\\ a+b=c^+\\ a-b=c^+\\ b-a=c^-}}\ \sum_{\substack{b,m\\ n+m=p^+\\ n-m=p^+\\ m-n=p^+}} \left(\frac{nmpc\pi^3}{4a(c^2\alpha^2+p^2\pi^2)}\right) c_{an}(t)\, c_{bm}(t)\\ &+\sum_{\substack{a,n\\ a+b=c^+\\ b-a=c^+\\ a-b=c^-}}\ \sum_{\substack{b,m\\ n-m=p^+\\ m-n=p^+\\ m+n=p^-}} \left(\frac{m^2pc\pi^3}{4b(c^2\alpha^2+p^2\pi^2)}\right) c_{an}(t)\, c_{bm}(t)\\ &-\sum_{\substack{a,n\\ a-b=c^+\\ b-a=c^+\\ a+b=c^-}}\ \sum_{\substack{b,m\\ n-m=p^+\\ m+n=p^+\\ n-m=p^-}} \left(\frac{nbc^2\alpha^2\pi}{4a(c^2\alpha^2+p^2\pi^2)}\right) c_{an}(t)\, c_{bm}(t)\\ &-\sum_{\substack{a,n\\ a-b=c^+\\ b-a=c^+\\ a+b=c^+}}\ \sum_{\substack{b,m\\ n-m=p^+\\ n+m=p^+\\ m-n=p^-}} \left(\frac{mc^2\alpha^2\pi}{4(c^2\alpha^2+p^2\pi^2)}\right) c_{an}(t)\, c_{bm}(t)\\ &-Pr\,(c^2\alpha^2+p^2\pi^2)\,c_{cp}(t)+Pr\left(\frac{c^2\alpha^2}{c^2\alpha^2+p^2\pi^2}\right) d_{cp}(t)\\ &+f_{cp}(t), \end{aligned} \tag{2.6}$$

$$\begin{aligned}\frac{d}{dt}(d_{cp}(t)) = &-\sum_{\substack{a,n\\ a-b=c^+\\ b-a=c^+\\ a+b=c^-}}\ \sum_{\substack{b,m\\ n-m=p^+\\ m+n=p^+\\ n-m=p^-}} (n\pi b/4a)\, c_{an}(t)\, d_{bm}(t)\\ &-\sum_{\substack{a,n\\ a-b=c^+\\ b-a=c^+\\ a+b=c^+}}\ \sum_{\substack{b,m\\ n-m=p^+\\ n+m=p^+\\ m-n=p^-}} (\tfrac{1}{4}m\pi)\, c_{an}(t)\, d_{bm}(t)\\ &+Rc_{cp}(t)-(c^2\alpha^2+p^2\pi^2)\,d_{cp}(t), \end{aligned} \tag{2.7}$$

$$\begin{aligned}\frac{d}{dt}(d_{op}(t)) = &-\sum_{\substack{a,n\\ m-n=p^+\\ m+n=p^+\\ n-m=p^-}}\ \sum_{a,m} (\tfrac{1}{4}n\pi)\, c_{an}(t)\, d_{am}(t)\\ &-\sum_{\substack{a,n\\ m-n=p^+\\ m+n=p^+\\ n-m=p^-}}\ \sum_{a,m} (\tfrac{1}{4}m\pi)\, c_{an}(t)\, d_{am}(t)\\ &-(p^2\pi^2)\,d_{op}(t), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned}f_{cj}(t) = \left(\frac{2\alpha}{\pi}\right)\left(\frac{c^2\alpha^2}{c^2\alpha^2+p^2\pi^2}\right)\int_0^{2\pi/\alpha}\int_0^1 \Bigg[\left(\frac{\partial S_{xx}}{\partial x}+\frac{\partial S_{xz}}{\partial z}\right)\left(-\left(\frac{p\pi}{c\alpha}\right)\sin(c\alpha x)\cos(p\pi z)\right)\\ +\left(\frac{\partial S_{zx}}{\partial x}+\frac{\partial S_{zz}}{\partial z}\right)(\cos(c\alpha x)\sin(p\pi z))\Bigg]\,dx\,dz. \end{aligned} \tag{2.9}$$

Only those components of wavenumbers (a, n, b, m), in the multiple summations above are considered which satisfy the selection rule written below the summation sign. Also the superscript sign is the sign of that term when that particular selection rule is satisfied. For example, for $c = 1$, $p = 1$, $a = 1$, $b = 2$, $n = 1$, $m = 2$, the first term in (2.6) becomes

$$(-1) \times (+1) \times \frac{(1 \times 2 \times 1 \times 1 \times \pi^3)}{4 \times 1 \times (\alpha^2 + \pi^2)} c_{11}(t)\, c_{22}(t)$$

or

$$-(\tfrac{1}{2}\pi^3(\alpha^2 + \pi^2))\, c_{11}(t)\, c_{22}(t).$$

The random-forcing function $f_{cp}(t)$ given by equation (2.9) is Gaussian, as is the random shear-stress tensor S_{ij} to which it is linearly related. Integrating (2.9) by parts and using correlations for S_{ij} from (2.2) and boundary conditions, we obtain the mean and correlations for the Gaussian forcing as,

$$\left.\begin{aligned} &\langle f_{cp}(t)\rangle = 0, \\ &\langle f_{cp}(t_1) f_{gh}(t_2)\rangle = 2\theta \left(\frac{2\alpha}{\pi}\right)(c^2\alpha^2)\, \delta(t_1 - t_2)\, \delta_{cg}\, \delta_{ph}. \end{aligned}\right\} \tag{2.10}$$

Having derived the random evolution equations (2.7)–(2.9) for the Fourier amplitudes with accompanying statistics of the forcing [see (2.10)] we need the initial statistics of these Fourier amplitudes in order to complete the mathematical formulation of the problem. We describe the initial state of the fluid as motionless and in pseudo-thermal equilibrium† at some average temperature, $\overline{T}$. Next we consider the equations governing the time evolution of the Fourier amplitudes from some arbitrary instant for an isothermal, quiescent fluid layer, and then determine the steady state correlations among the Fourier amplitudes. Neglecting the nonlinear term in (2.6) for small fluctuations, the evolution equation for $c_{cp}(t)$, becomes:

$$\frac{d}{dt} c_{cp}(t) = -Pr\,(c^2\alpha^2 + p^2\pi^2)\, c_{cp}(t) + f_{cp}(t). \tag{2.11}$$

Since this is a linear Ito stochastic differential equation, $c_{cp}(t)$ is a Gaussian process with vanishing mean. Denoting $\langle c_{cp}(t)\, c_{gh}(t)\rangle$ as $E_{cpgh}(t)$ and $\langle f_{cp}(t) f_{gh}(t)\rangle$ as

$$2D_{cpgh}\, \delta(t_1 - t_2),$$

the equation for the second-order moment, $E_{cpgh}(t)$, may be readily written following Soong (1973), as,

$$\frac{d}{dt} E_{cpgh} = -Pr\,(c^2\alpha^2 + p^2\pi^2)\, E_{cpgh} - Pr\,(g^2\alpha^2 + h^2\pi^2)\, E_{ghcp} + D_{cpgh} + D_{ghcp}, \tag{2.12}$$

where

$$D_{cpgh} = \theta \left(\frac{2\alpha}{\pi}\right)(c^2\alpha^2)\, \delta_{cg}\, \delta_{ph}. \tag{2.13}$$

The steady state solution, readily obtained from above is

$$E_{cpgh} = \theta \left(\frac{2\alpha}{\pi}\right)\left(\frac{1}{Pr}\right)\left(\frac{c^2\alpha^2}{c^2\alpha^2 + p^2\pi^2}\right) \delta_{cg}\, \delta_{ph}. \tag{2.14}$$

† The concept of pseudo-thermal equilibrium implies that each infinitesimal horizontal fluid layer is insulated from the adjacent ones.

In summary, the Fourier amplitudes, c_{cp}, are initially Gaussian distributed with vanishing mean and correlations given by (2.14) while the Fourier amplitudes, d_{cp}, have trivial initial values.

(c) *Moment equations*

Since the random forces defined by equations (2.9–2.10) are δ-correlated Gaussian processes, equations (2.6–2.8) constitute a system of quadratically nonlinear Ito stochastic differential equations (Soong 1973). Replacing the double subscripts (c, p) by a single subscript (i), these coupled equations can be written as:

$$\frac{dX_i}{dt} = \sum_{j=1}^{n} a_{ij} X_j + \sum_{j=1}^{n} \sum_{k=j}^{n} Q_{ijk} X_j X_k + W_i(t), \tag{2.15}$$

where the random variables $\{X_1, X_2, \ldots, X_n\}$ and the random forces $\{W_1, W_2, \ldots W_n\}$, are respectively the Fourier amplitudes $\{c_{cp}, d_{cp}\}$ and the random forces $\{f_{cp}\}$. Here we have truncated the infinitely coupled hierarchy to just n equations. The value of the cutoff index n depends on the convergence criteria and is discussed later. The correlation among random forces from (2.10), in equivalent notation is then,

$$\langle W_i(t_1)\, W_j(t_2)\rangle = 2D_{ij}\,\delta(t_1 - t_2). \tag{2.16}$$

Here D_{ij} is a diagonal array obtained from (2.10), and the a_{ij} and Q_{ijk} are inner-product arrays which are known. Denoting the nth-order moment $\langle X_{i_1} X_{i_2} \ldots X_{i_n}\rangle$, by $E_{i_1 i_2 \cdots i_n}$, the hierarchy of moment equations for system can readily be written following Soong, as:

$$\frac{dE_r}{dt} = \sum_{j=1}^{n} a_{rj} E_j + \sum_{j=1}^{n} \sum_{k=j}^{n} Q_{rjk} E_{jk} \quad r = 1, \ldots n, \tag{2.17}$$

$$\frac{dE_{rm}}{dt} = \sum_{j=1}^{n} (a_{rj} E_{jm} + a_{mj} E_{jr}) + \sum_{j=1}^{n} \sum_{k=j}^{n} (Q_{rjk} E_{jkm} + Q_{mjk} E_{jkr}) + (D_{rm} + D_{mr}) \quad \begin{cases} r = 1, \ldots n, \\ m = r, \ldots n, \end{cases} \tag{2.18}$$

$$\frac{dE_{rmq}}{dt} = \sum_{j=1}^{n} (a_{rj} E_{jmq} + a_{mj} E_{jrq} + a_{qj} E_{jrm}) + \sum_{j=1}^{n} \sum_{k=j}^{n} (Q_{rjk} E_{jkmq} + Q_{mjk} E_{jkrq} + Q_{qjk} E_{jkrm}) + (D_{rm} + D_{mr}) E_q + (D_{qr} + D_{rq}) E_m + (D_{mq} + D_{qm}) E_r \quad \begin{cases} r = 1, \ldots n, \\ m = r, \ldots n, \\ q = m, \ldots n. \end{cases} \tag{2.19}$$

Higher-order moment equations can easily be written by a way of generalization. We note that the nth-order moment equation contains moment terms of nth, $(n+1)$ and $(n-2)$ order. That the moment equations are not closed is a general property of nonlinear random differential equations.

3. Linear analysis

The linear instability theory for the random problem was solved exactly by Zaitsev & Shliomis (1971), who were able to show, among other things, that the critical point is that given by the solution to a standard eigenvalue problem, and that correlation lengths and relaxation times diverge like $(R-R_c)^{-\frac{1}{2}}$ and $(R-R_c)^{-1}$, respectively, as R approaches the critical point (R_c). It is useful to briefly develop these results in a more explicit form here, for use in later sections.

The linear equations are

$$\frac{d}{dt}(c_{cp}(t)) = -Pr\,(c^2\alpha^2+p^2\pi^2)c_{cp}(t)+Pr\left(\frac{c^2\alpha^2}{c^2\alpha^2+p^2\pi^2}\right)d_{cp}(t)+f_{cp}(t), \tag{3.1}$$

$$\frac{d}{dt}(d_{cp}(t)) = Rc_{cp}(t)-(c^2\alpha^2+p^2\pi^2)\,d_{cp}(t), \tag{3.2}$$

$$\frac{d}{dt}(d_{op}(t)) = -(p^2\pi^2)\,d_{op}(t), \tag{3.3}$$

with initial conditions

$$E_{cpgh}(0) = \langle c_{cp}(t)\,c_{gh}(t)\rangle_{t=0} = \frac{\theta}{Pr}\left(\frac{2\alpha}{\pi}\right)\left(\frac{c^2\alpha^2}{c^2\alpha^2+p^2\pi^2}\right)\delta_{cg}\,\delta_{ph}, \tag{3.4}$$

$$H_{cpgh}(0) = \langle c_{cp}(t)\,d_{gh}(t)\rangle_{t=0} = 0, \tag{3.5}$$

$$G_{cpgh}(0) = \langle d_{cp}(t)\,d_{gh}(t)\rangle_{t=0} = 0, \tag{3.6}$$

and the correlation of random forces given by (2.10). Note that we denote $\langle c_{cp}\,d_{gh}\rangle$ as H_{cpgh}.

Since the equation for $d_{op}(t)$ is uncoupled, homogeneous and with trivial initial condition, it has a trivial solution. Also the correlation of two Fourier amplitudes with different horizontal or vertical wavenumbers ($c \neq g$ or $p \neq h$) reduces to the trivial solution because of the $\delta_{cg}\,\delta_{ph}$ dependence of initial conditions and random force correlations. Non-trivial second-order moment equations similar to (2.18) for the stochastic equations (2.20) and (2.21) are

$$\frac{d}{dt}(E_{cpcp}) = -2\,Pr\,(c^2\alpha^2+p^2\pi^2)\,E_{cpcp}+2\,Pr\left(\frac{c^2\alpha^2}{c^2\alpha^2+p^2\pi^2}\right)H_{cpcp}+2D_{cpcp}, \tag{3.7}$$

$$\frac{d}{dt}(H_{cpcp}) = RE_{cpcp}-(c^2\alpha^2+p^2\pi^2)\,(1+Pr)\,H_{cpcp}+Pr\left(\frac{c^2\alpha^2}{c^2\alpha^2+p^2\pi^2}\right)G_{cpcp}, \tag{3.8}$$

$$\frac{d}{dt}(G_{cpcp}) = 2RH_{cpcp}-2(c^2\alpha^2+p^2\pi^2)\,G_{cpcp}, \tag{3.9}$$

where, from (2.14)

$$D_{cpcp} = \theta\left(\frac{2\alpha}{\pi}\right)(c^2\alpha^2).$$

Letting

$$\mathbf{X}_{cp} = [E_{cpcp},\ H_{cpcp},\ G_{cpcp}]^{\mathrm{T}}, \tag{3.10}$$

$$\mathbf{I}_{cp} = [2D_{cpcp}, 0, 0]^{\mathrm{T}} \tag{3.11}$$

and

$$\mathbf{A}_{cp} = \begin{bmatrix} -2Pr\,(c^2\alpha^2+p^2\pi^2), & \dfrac{2Pr\,c^2\alpha^2}{c^2\alpha^2+p^2\pi^2}, & 0 \\ R, & -(c^2\alpha^2+p^2\pi^2)\,(1+Pr), & \dfrac{Pr\,c^2\alpha^2}{(c^2\alpha^2+p^2\pi^2)} \\ 0, & 2R, & -2(c^2\alpha^2+p^2) \end{bmatrix} \tag{3.12}$$

the above equations can be written in matrix form as,

$$\frac{d}{dt}(\mathbf{X}_{cp}) = \mathbf{A}_{cp}\,\mathbf{X}_{cp} + \mathbf{I}_{cp} \tag{3.13}$$

with

$$\mathbf{X}^0_{cp} = \mathbf{X}_{cp}(t=0) = [E_{cpcp}(t=0), \quad 0, \quad 0]^{\mathrm{T}}. \tag{3.14}$$

The solution of (3.13) with initial condition (3.14) can be written explicitly (Coddington & Levinson 1955) as

$$\mathbf{X}_{cp}(t) = e^{(\mathbf{A}_{cp}t)}(\mathbf{X}^0_{cp} + \mathbf{A}^{-1}_{cp}\,\mathbf{I}_{cp}) - \mathbf{A}^{-1}_{cp}\,\mathbf{I}_{cp}, \tag{3.15}$$

where

$$e^{(\mathbf{A}_{cp}t)} = \mathbf{P}e^{t\mathbf{J}}\mathbf{P}^{-1}. \tag{3.16}$$

Here $\mathbf{J}$ is the Jordan–Canonical form for the matrix $\mathbf{A}_{cp}$ and $\mathbf{P}$ is the matrix of eigenvectors such that $\mathbf{AP} = \mathbf{PJ}$. Since all the eigenvalues of $\mathbf{A}_{cp}$ are distinct, matrix $\mathbf{J}$ has a simple diagonal representation

$$\mathbf{J} = \begin{pmatrix} \lambda^{(1)}_{cp} & 0 & 0 \\ 0 & \lambda^{(2)}_{cp} & 0 \\ 0 & 0 & \lambda^{(3)}_{cp} \end{pmatrix}, \tag{3.17}$$

where $\lambda^{(1)}_{cp}$, $\lambda^{(2)}_{cp}$ and $\lambda^{(3)}_{cp}$ are eigenvalues of the matrix $\mathbf{A}_{cp}$ (cf. appendix for the eigenvalues and eigenvectors of the matrix $\mathbf{A}_{cp}$); $e^{\mathbf{J}t}$ is then simply

$$e^{\mathbf{J}t} = \exp\begin{pmatrix} \lambda^{(1)}_{cp}\,t & 0 & 0 \\ 0 & \lambda^{(2)}_{cp}\,t & 0 \\ 0 & 0 & \lambda^{(3)}_{cp}\,t \end{pmatrix}. \tag{3.18}$$

Two of the eigenvalues $\lambda^{(2)}_{cp}$ and $\lambda^{(3)}_{cp}$ are always negative; however, $\lambda^{(1)}_{cp}$ goes through zero for critical values of Rayleigh number, defined as $R_{cp}(\alpha) = (c^2\alpha^2+p^2\pi^2)^3/c^2\alpha^2$. As is well known, the lowest value of R_{cp} for $c=1$, $p=1$ goes through a minimum value ($R^*_{11} = 27\pi^4/4$) for $\alpha^* = \pi/\sqrt{2}$. Thus for $R < R^*_{11}$ all eigenvalues are negative and thermodynamic fluctuations relax to steady state values, i.e.

$$\lim_{t\to\infty} \mathbf{X}_{cp}(t) = -\mathbf{A}^{-1}_{cp}\,\mathbf{I}_{cp}.$$

For $R > R^*_{11}$ at least one eigenvalue is positive (for a range of α) and correlations grows exponentially in time. Eventually, the assumptions of linear theory become invalid and nonlinear terms have to be taken into account.

4. Weak nonlinear theory

(*a*) *Order analysis of nonlinear equations*

We restrict ourselves to weakly supercritical conditions for which $R > R^*_{11}$ but $R < R_{cp}(\alpha)$, $c > 1$, $p > 1$ so that only $\lambda^{(1)}_{11} > 0$, $\lambda^{(1)}_{cp} < 0$ for $c > 1, p > 1$. The condition

$\lambda_{11}^{(1)} > 0$ restricts the value of α to a small range near α_c. The analysis that follows is for arbitrary but fixed wavenumber. For R close to R_{11}^*, in the linear analysis, only the correlations $\langle c_{11}c_{11}\rangle$, $\langle c_{11}d_{11}\rangle$, and $\langle d_{11}d_{11}\rangle$ grow exponentially with time while all other correlations remain of thermodynamic order. As correlations grow to $O(1)$ the nonlinear terms also grow to $O(1)$ and have to be retained in the analysis. Consider the nonlinear stochastic modal equations (2.6)–(2.8). We see that only those Fourier amplitudes (c_{cp}, d_{cp}), generated by nonlinear interaction of modes c_{11}, d_{11} grow to $O(1)$ when $(c+p)$ is even, because of the particular nature of selection rule generating them. For example, generation of the mean term d_{02} in (2.8) results from the direct interaction of c_{11} and d_{11}. Higher-order interaction generates terms like c_{13}, d_{13} from c_{11} and d_{02}. We thus separate the nonlinear equations for the evolution of Fourier amplitudes into two parts; those which evolve to $O(1)$ ($c+p$ even), and those which stay of thermodynamic order (i.e. of order θ), ($c+p$ odd). We first consider those modes which grow to $O(1)$.

Since the number of terms generated by nonlinear interaction of c_{11} and d_{11} rapidly inflates, we restrict the analysis to the lowest non-trivial nonlinear interaction, *viz.* the generation of mean term d_{02}. This may be justified in the usual way using perturbation theory about $R = R_c$ (Malkus & Veronis 1958). Denoting the Fourier amplitudes (c_{11}, d_{11}, d_{02}) as random variables (X_1, X_2, X_3) and the random force $f_{11}(t)$ as $f_1(t)$, the nonlinear stochastic equations may be written as,

$$\frac{dX_1}{dt} = a_{11}X_1 + a_{12}X_2 + f_1(t), \tag{4.1a}$$

$$\frac{dX_2}{dt} = a_{21}X_1 + a_{22}X_2 + Q_{213}X_1X_3, \tag{4.1b}$$

$$\frac{dX_3}{dt} = a_{33}X_3 + Q_{312}X_1X_2, \tag{4.1c}$$

where

$$\left.\begin{aligned} a_{11} &= -Pr\,(\alpha^2+\pi^2), \quad a_{12} = \frac{Pr\,\alpha^2}{\alpha^2+\pi^2}, \quad a_{21} = R, \\ a_{22} &= -(\alpha^2+\pi^2), \quad a_{33} = -4\pi^2, \quad Q_{213} = \pi, \quad Q_{312} = -\tfrac{1}{2}\pi, \end{aligned}\right\} \tag{4.2}$$

with $\langle f_1(t_1)f_1(t_2)\rangle = 2D_{11}\,\delta(t_1-t_2)$, $D_{11} = 2\alpha^3\theta/\pi$.

(*b*) *The associated Fockker–Planck equation and the steady solution*

We consider the hierarchy of moment equations for the stochastic system (4.1) and the associated Fockker–Planck equation for the probability density function $f(x_1, x_2, x_3, t)$ (Soong 1973):

$$\left.\begin{aligned} \frac{\partial f}{\partial t} = &-\frac{\partial}{\partial x_1}\{(a_{11}x_1 + a_{12}x_2)f\} - \frac{\partial}{\partial x_2}\{(a_{21}x_1 + a_{22}x_2 \\ &+ Q_{213}x_1x_3)f\} - \frac{\partial}{\partial x_3}\{(a_{33}x_3 + Q_{312}x_1x_2)f\} + D_{11}\frac{\partial^2 f}{\partial x_1^2}. \end{aligned}\right\} \tag{4.3}$$

An important property of the moments of the solution can be derived using symmetry of the Fockker–Planck equation. It can be easily verified that the Fockker–Planck

equation is invariant under the transformation $(x_1, x_2, x_3) \to (-x_1, -x_2, x_3)$, implying,

$$f(x_1, x_2, x_3) = f(-x_1, -x_2, x_3). \tag{4.4}$$

Next consider the moments $\langle X_1^{(K_1)} X_2^{(K_2)} X_3^{(K_3)} \rangle$ where K_1, K_2, and K_3 are integers.

$$\langle X_1^{(K_1)} X_2^{(K_2)} X_3^{(K_3)} \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1^{K_1} x_2^{K_2} x_3^{K_3} f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3. \tag{4.5}$$

Using the invariance properly (4.4) it is possible to show that

$$\langle X_1^{(K_1)} X_2^{(K_2)} X_3^{(K_3)} \rangle = 2\{(-1)^{(K_1+K_2)} + 1\} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} x_1^{K_1} x_2^{K_2} x_3^{K_3} f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3. \tag{4.6}$$

Hence

$$\langle X_1^{(K_1)} X_2^{(K_2)} X_3^{(K_3)} \rangle = 0, \quad (K_1 + K_2) = \text{odd integer}.$$

Using this property, denoting $\langle X_i X_j X_k \dots X_l \rangle$ as $E_{ijk\dots l}$ we write the hierarchy of moment equations for the non-trivial moments, from (2.17)–(2.19)

$$\left.\begin{aligned}
&\frac{dE_3}{dt} = a_{33} E_3 + Q_{312} E_{12}, \quad \frac{dE_{11}}{dt} = 2a_{11} E_{11} + 2a_{12} E_{12} + 2D_{11}, \\
&\frac{dE_{12}}{dt} = a_{21} E_{11} + (a_{11} + a_{22}) E_{12} + a_{12} E_{22} + Q_{213} E_{113}, \\
&\frac{dE_{22}}{dt} = 2a_{22} E_{22} + 2a_{21} E_{12} + 2Q_{213} E_{123}, \quad \frac{d}{dt} E_{33} = 2a_{33} E_{33} + 2Q_{312} E_{123}, \\
&\frac{d}{dt} E_{113} = (2a_{11} + a_{33}) E_{113} + 2a_{12} E_{123} + Q_{312} E_{1112} + 2D_{11} E_3, \\
&\frac{d}{dt} E_{123} = (a_{11} + a_{22} + a_{33}) E_{123} + a_{12} E_{223} + a_{21} E_{113} + Q_{213} E_{1133} + Q_{312} E_{1122}, \\
&\frac{d}{dt} E_{223} = (2a_{22} + a_{33}) E_{223} + 2a_{21} E_{123} + 2Q_{213} E_{1233} + Q_{312} E_{1222}, \\
&\frac{d}{dt} E_{333} = 3a_{33} E_{333} + 3Q_{312} E_{1233},
\end{aligned}\right\} \tag{4.7}$$

with higher-order moment equations written in a similar fashion.

As $t \to \infty$, the moments grow to $O(1)$ and terms multiplied by D_{11} in (4.7) become negligible. Time independent solutions, if they exist, can thus be found by setting d/dt and D_{11} to zero and solving the resulting set of infinite coupled algebraic moment equations. However the moment equations are equivalent to the Fockker–Planck equation, and another alternative is to solve for the steady probability density function $f_s(x_1, x_2, x_3)$ from (4.3) without the D_{11} term. Instead of trying to solve the partial differential equation for f_s, we construct the steady probability distribution function as follows. Since the importance of random forcing vanishes at larger times, we can set the term $f_1(t)$ to zero in equation (4.1). Then the randomness enters into the equation through virtual initial conditions and each sample trajectory evolves and stabilizes deterministically. Then to obtain the steady state distribution function for the random vector $X_i(t = \infty)$, we can set the time derivative (d/dt) to zero in equation

(4.1). This reduces equation (4.1) to algebraic equations, whose solution can be written in a straightforward manner for the stationary random variables (X_{1S}, X_{2S}, X_{3S}):

$$\left.\begin{aligned} X_{1S} &= \pm\left[\frac{a_{33}a_{12}}{Q_{312}a_{11}}\left(\frac{a_{11}a_{22}-a_{21}a_{12}}{a_{12}Q_{213}}\right)\right]^{\frac{1}{2}} = \pm\beta \quad \text{(say)}, \\ X_{2S} &= -\left(\frac{a_{11}}{a_{12}}\right)X_1, \quad X_{3S} = \frac{a_{11}a_{22}-a_{21}a_{12}}{a_{12}Q_{213}} = \gamma \quad \text{(say)}. \end{aligned}\right\} \tag{4.8}$$

X_{1S} takes the value $\pm\beta$ each with probability $\frac{1}{2}$; X_{2S} is linearly related to X_{1S}; X_{3S} is statistically independent of X_{1S} and X_{2S}, and takes the value γ with probability 1. Or in an equivalent form,

$$f_S(x_1, x_2, x_3) = \{\tfrac{1}{2}\delta(x_1-\beta)\,\delta(x_2+a_{11}\beta/a_{12}) + \tfrac{1}{2}\delta(x_1+\beta)\,\delta(x_2-a_{11}\beta/a_{12})\}\,\delta(x_3-\gamma). \tag{4.9}$$

Next we calculate the steady moments $\langle X_{1S}^{(K_1)} X_{2S}^{(K_2)} X_{3S}^{(K_3)}\rangle$.

$$\begin{aligned} \langle x_{1S}^{(K_1)} x_{2S}^{(K_2)} x_{3S}^{(K_3)}\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^{K_1} x_2^{K_2} x_3^{K_3} f_S(x_1, x_2, x_3)\,dx_1\,dx_2\,dx_3 \\ &= \tfrac{1}{2}\{(-1)^{(K_1+K_2)}+1\}\,\beta^{(K_1)}\left(\frac{-a_{11}\beta}{a_{11}}\right)^{(K_2)}\gamma^{(K_3)}. \end{aligned}$$

Hence

$$\begin{aligned} \langle X_{1S}^{(K_1)} X_{2S}^{(K_2)} X_{3S}^{(K_3)}\rangle &= \beta^{(K_1)}\left(\frac{-a_{11}\beta}{a_{12}}\right)^{(K_2)}\gamma^{(K_3)}, \quad (K_1+K_2) = \text{even integer} \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{4.10}$$

The steady solution so obtained is identical to the deterministic two-dimensional roll cell solution of Malkus & Veronis (1958) to lowest order. Superimposed on this finite amplitude roll cell are thermodynamic fluctuations which we had described previously in terms of Fourier amplitudes (c_{cp}, d_{cp}), $(c+p)$ odd. One can solve for these amplitudes in order to correctly represent the effect of thermodynamic fluctuations on roll cells; however, we do not do so here since our main interest is to show the growth of observable motion from thermodynamic fluctuations.

5. Time evolution

As mentioned earlier, in the time-dependent solution process, we are looking for the evolution of statistical mean value properties of the ensemble; one of the most important of these is the heat transport, $\langle Nu(t)\rangle$. It can be easily shown that,

$$\langle Nu\,(t)\rangle = 1 - \frac{2\pi}{R}E_3(t). \tag{5.1}$$

The time-dependent moment properties may be calculated from the probability density function $f(x_1, x_2, x_3, t)$, if one can solve the Fockker–Planck equation (4.3). For small supercriticality, the non-linearity is expressible as a gradient field and the steady state solution of a Fockker–Planck equation such as (4.3) is represented in terms of a potential (Ludwig 1975; Graham 1974). Then the time-dependent solution

for the probability density function can be obtained in an approximate way by an asymptotic ray method developed by Ludwig (1975). Since our interest here is to develop a numerical method for solutions at high Rayleigh numbers, we do not utilize such analytical techniques in what follows.

In order to obtain the time-dependent moment solution, one has to integrate the infinite hierarchy of moment equations from their initial values. However, some closure approximation is necessary to truncate this hierarchy. We discuss this closure problem below. A second alternative is to simulate a large number of realizations,

$$\{x_2^1(t), x_2^2(t), \ldots, x_i^N(t)\},$$

approximating the ensemble of the random evolution $X_i(t)$; and then to calculate the moments from this approximate ensemble. We follow this second alternative and generate the approximate ensemble by a Monte Carlo simulation. With this simulation, we can also study the evolution of individual realizations.

The white-noise component $f_1(t)$ in (4.1) may be formally written as a derivative, $dB(t)/dt$, where $B(t)$ is the Weiner process or the Brownian motion process, Gaussian distributed with statistics:

$$\left.\begin{aligned} \langle B(t)\rangle &= 0, \\ \langle B(t_1)\,B(t_2)\rangle &= 2D_{11}\min(t_1, t_2) \end{aligned}\right\} \tag{5.2}$$

(Soong 1973). Following Chandrasekhar (1943), we then integrate (4.1) for small $(t_2 - t_1)$ as

$$X_i(t_2) = X_i(t_1) + \int_{t_1}^{t_2} (a_{ij}X_j + Q_{ijk}X_j X_k)\,dt + (B_i(t_2) - B_i(t_1)). \tag{5.3}$$

The integral appearing in (5.3) is deterministic, while the increment due to the random forcing is included in the term $B_i(t_2) - B_i(t_1)$. The increment $B_i(t_2) - B_i(t_1)$ is then Gaussian distributed with statistics

$$\left.\begin{aligned} \langle B_i(t_2) - B_i(t_1)\rangle &= 0, \\ \langle (B_i(t_2) - B_i(t_1))^2\rangle &= 2D_{ii}|t_2 - t_1| = 2D_{ii}h, \end{aligned}\right\} \tag{5.4}$$

where h is the integration step size. A Monte Carlo simulation for the solution process $X_i(t)$ then consists of following steps. First we generate N vectors $\{x_i^1(0), x_i^2(0), \ldots, x_i^N(0)\}$ as the realizable initial values for $x_i(0)$. They are Gaussian distributed with mean and variance specified in § B 2 and are generated numerically using a polar method of pseudo-random number generation (Knuth 1969). For each time step, these realizations are first deterministically integrated using standard initial value techniques. Next, N realizations for the random increment $B_i(t_2) - B_i(t_1)$, Gaussian distributed with statistics (5.4), are generated using the same method of pseudo-random number generation. They are added to the deterministic increment to complete the step integration. Integrating all realizations until they reach a steady state completes the simulation.

For the numerical calculations we use the following values for the parameters R, Pr, θ and α. We choose $Pr = 7$, as being typical for liquids. Since $(k\bar{T}/\rho\nu^2 l) = 10^{-9}$ for $l \approx 0{\cdot}1$ cm, $\theta = Pr^3 \times 10^{-9} = 3{\cdot}43 \times 10^{-7}$. For small $R - R_c$, convection sets in with α close to $\alpha^* = \pi/\sqrt{2}$. Graham (1974) has considered the problem with weak random

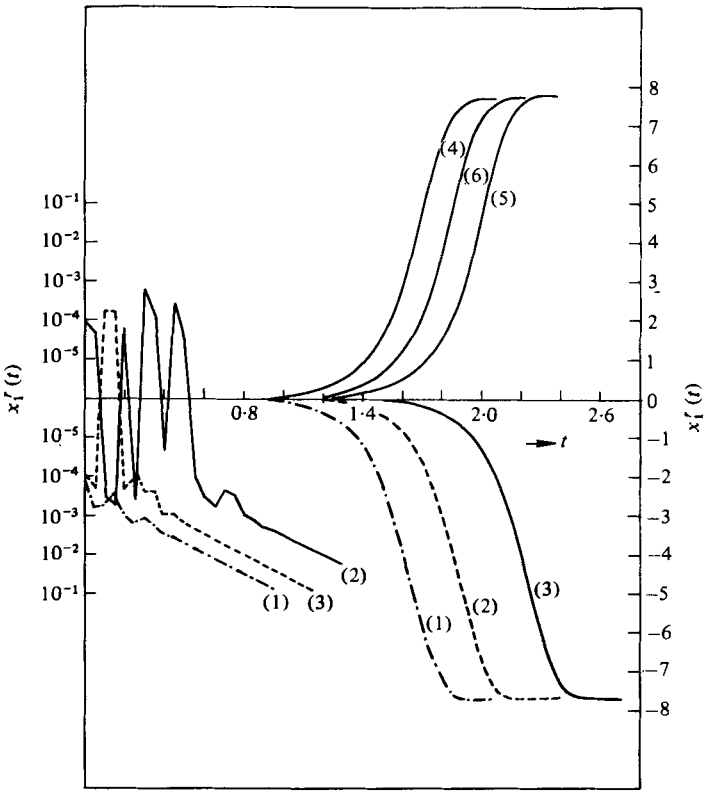

FIGURE 1. A few typical realizations, $x_1^r(t)$, from the Monte Carlo simulation. The conditions are given in the text.

perturbations in $(\alpha - \alpha^*)$, and has shown that the roll that evolves has $\alpha = \alpha^*$. Thus we study the evolution for this critical value of wavenumber. Then

$$R_{11}(\alpha^*) = R_{11}^* = 27\pi^4/4,$$

$R_{12}(\alpha^*) = 27R_{11}^*$, $R_{21}(\alpha^*) = 2R_{11}^*$, and $R_{22}(\alpha^*) = 16R_{11}^*$. As mentioned above, the value of R is therefore restricted to $R_{11}^* < R < 2R_{11}^*$.

The choice of N as the number of realizations necessary to generate a reasonably good distribution for $x_i(t)$, is rather intricate. Consider a random variable distributed with mean and variance μ and σ^2 respectively. If one generates N sample realizations for this random variable, then the mean calculated from these samples is distributed with variance σ^2/N. Hence the error involved in approximating the sample mean as the ensemble mean is of order $\sigma/\sqrt{N}$. For a Gaussian random variable with zero mean and variance σ^2, $\langle x^4\rangle - \langle x^2\rangle = 2\sigma^4$. The variance σ_N^2 of N independent realizations of x, is then distributed with variance $2\sigma^4/N$. We found that for $N = 30$, the error involved in assuming $\sigma_N^2 = \sigma^2$ is $(2/30)^{\frac{1}{2}}\sigma^2$; roughly 7 %. That is to say that the sample variance σ_N^2 may be somewhere between $0{\cdot}93\sigma^2$ to $1{\cdot}07\sigma^2$, with probability concentrated near σ^2. This is the minimum value of N we have used during simulation; however some simulations were generated using N as large as 200.

The first component of the vector $x_i(t)$ represents the amplitude of the convective flow. We plot few typical realizations for $x_i(t)$ for the case $R = 1{\cdot}5R^*$ in figure 1. For

good resolution, two different scales are used for the vertical axis. The vertical axis in the right-hand side of the figure is scaled linearly and corresponds to the evolution at larger times. The vertical axis on the left-hand side of the figure is scaled logarithmically and corresponds to the evolution at smaller times. Curves 1 and 2 are two extremes of observed trajectories stabilizing to roll cells with a negative flow direction. Curve 3 represents an intermediate case. Similarly curves 4, 5, and 6 are respectively two extremes and an intermediate sample stabilizing to roll cells with positive flow direction. In view of the vanishing mean of the random forcing, half of the realizations should stabilize to roll cells with positive flow direction and the other half to negative flow direction. Owing to inherent errors in the Monte Carlo simulation, this distribution was found to be close to that expected with 14 realizations stabilizing to positive flow direction and 16 to negative.

Three distinct stages during evolution are apparent. There is an initial stage wherein realizations are of the same order as the forcing, which has a substantial influence in deciding the fate of any given sample at later times. For instance, sample 1 stays for a minimum time under the influence of forcing and thus represents the fastest evolving realization. Sample 2 on the contrary stays under the influence of forcing for a long interval and thus evolves slowest of all. It also has a positive initial velocity but stabilizes to a roll cell with negative flow direction. Once the realizations leave the influence of forcing, their trajectory is decided. Thus follows an intermediate state during which realizations evolve in a deterministic fashion without any significant influence due to forcing. However, nonlinear interactions are negligible during this stage as evidenced by exponential growth of realizations and also by Nusselt numbers which remains unity during this stage. At the end of this second stage, realizations have grown to such a magnitude that nonlinear interactions become substantial. Realizations then enter the final stage of evolution wherein nonlinear effects dominate and modify the growth of realization till they reach a steady state.

Such different regions of growth appear in previous studies of nonlinear Brownian motion (Suzuki 1978). Suzuki has considered one nonlinear stochastic equation of the form

$$dx/dt = ax - bx^2 + \epsilon f(t). \tag{5.5}$$

For a small strength of forcing $\epsilon \ll 1$, the method of multiple scales may then be utilized to construct an analytical solution. He has formulated this method in terms of a scaling theory of transient nonlinear fluctuations. However, extension of his theory to the case of high R is impossible: we use the agreement between our numerical results and the general qualitative features of the evolution of a single randomly-forced mode to establish some confidence in our simulation scheme.

The time evolution of the heat transport for each of the samples as well as the mean is of interest, as it gives one of the primary measurable quantities. This is shown in figure 2, where we plot the Nusselt number for each of the realizations as well as their mean. The conditions and samples are identical to those of figure 1. Two features are immediately apparent from the figure. The first is that different cells begin to convect at different times, and so there is a distribution of times at which the members of the population reach a given convective strength. The mean over this distribution gives the transition time. The second feature of the results is that the mean does not follow the same trajectory as the individual samples. This is a result of the fact that in the

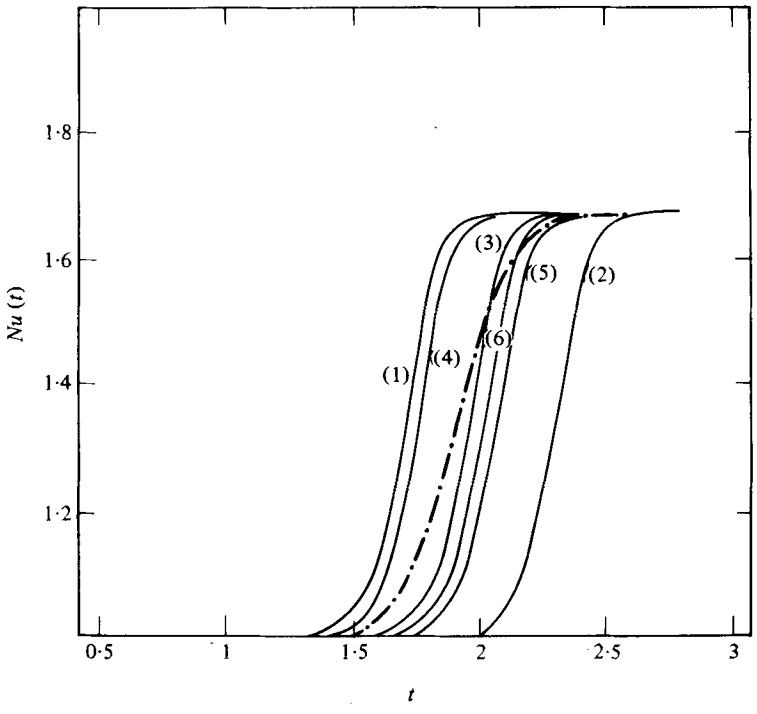

FIGURE 2. Nusselt number *vs.* time for the realizations of figure 1 (——) and the mean (—·—·).

nonlinear regime, the probability density contribution is evolving from Gaussian to its steady form given by (4.9). Interestingly, as we will show below, the mean convective transport $Nu(t)$ does *not* follow a deterministic trajectory, although each of the samples does.

We now return to the method of moment truncation. The infinite hierarchy of moment equations (4.7) may be closed by making some approximation. Schemes range from simply dropping the higher order moments at a certain truncation point to approximating them in terms of lower-order moments in some optimal way. At steady state this functional dependence of higher-order moments may be established from (4.10) at third order, *viz.*

$$E_{113} = E_{11}E_3, \quad E_{123} = E_{12}E_3. \tag{5.6}$$

The advantage of using this truncation is that the solution will evolve to correct steady state values. The first five of equations (4.7), together with the closure hypothesis (5.6), form a closed system of moment equations. We plot $\langle Nu(t)\rangle$ from these moment quations for three different values of Rayleigh number in figure 3, i.e. for $R = 1{\cdot}3R_c$, $1{\cdot}5R_c$, and $1{\cdot}8R_c$. In the same figure we plot the evolutions from the Monte Carlo simulation. At small times, when nonlinearities are unimportant, the truncation scheme does not affect the evolution and the solution obtained from moment truncation matches with that obtained from Monte Carlo simulation. However, in the nonlinear regime, we see that the magnitude predicted by the moment truncation method is overestimated. To understand this, we study the evolution of $E_{113}/E_{11}E_3$ and

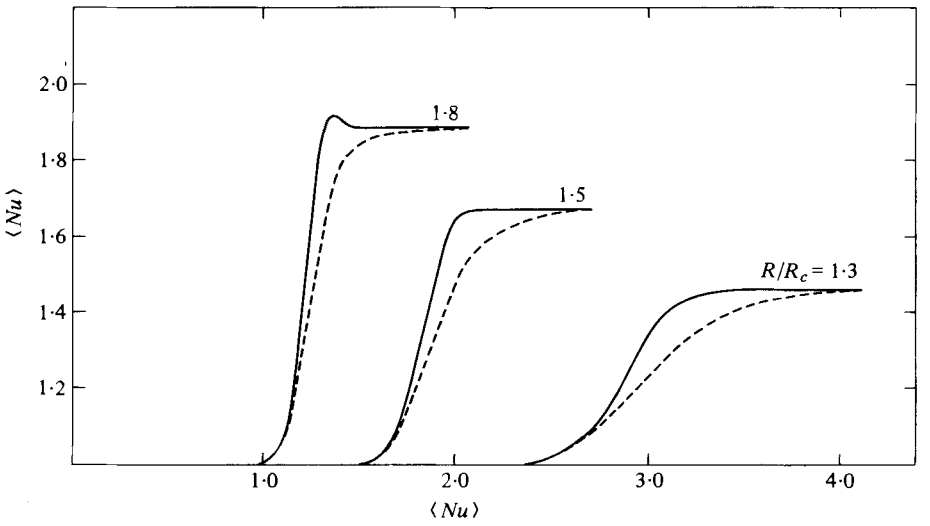

FIGURE 3. The heat transport during evolution: ——, moment truncation; – – – –, Monte Carlo.

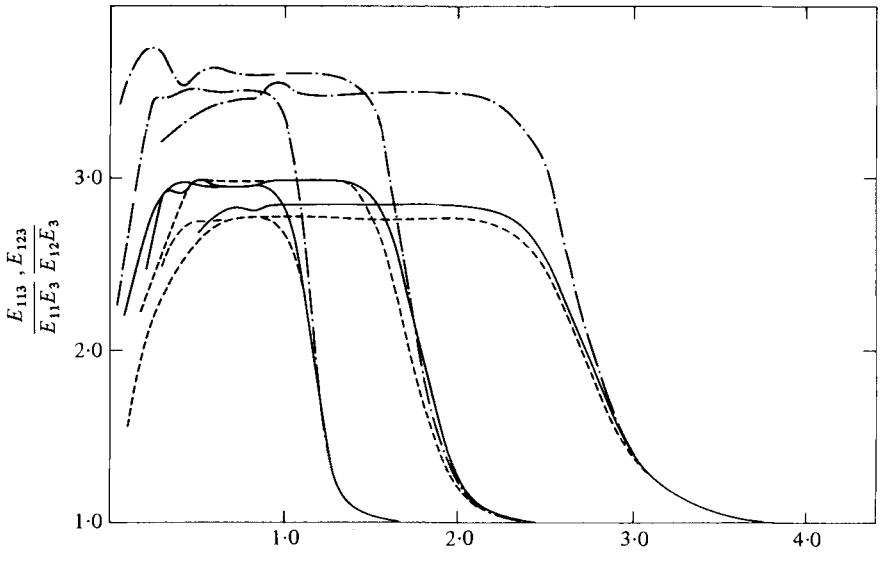

FIGURE 4. The moment ratios $E_{113}/E_{11}E_3$ and $E_{123}/E_{123}E_3$ from the Monte Carlo simulations. ——, $N = 200$, $h = 0{\cdot}005$; – – – –, $N = 100$, $h = 0{\cdot}01$; —·—·, $N = 100$, $h = 0{\cdot}005$.

$E_{123}/E_{12}E_3$ as determined from the Monte Carlo method; refer to figure 4. These ratios, which are taken as 1 from the moment truncation, in fact evolve from a value close to 3, to 1 during evolution.

The functional dependence of the third-order moment at small times may be established as follows. During the intermediate region, both random forcing and nonlinear effects have negligible influence. Then the terms $f_1(t)$ and $Q_{213}x_1x_3$ may be neglected in (4.1) and thus (4.1*c*) may be uncoupled from (4.1*a*, *b*). Setting x_{10} be the

value for x_1 at any origin of time t_0, in the second stage, the solution for (4.1) may be written as:†

$$\left.\begin{aligned} x_1 &= k_1 x_{10} e^{\lambda_1(t-t_0)}, \\ x_2 &= k_2 x_{10} e^{\lambda_1(t-t_0)}, \\ x_3 &= k_3 x_{10}^2 e^{2\lambda_1(t-t_0)}, \end{aligned}\right\} \tag{5.7}$$

where k_1, k_2, and k_3 are constants. Then

$$\frac{E_{113}}{E_{11}E_3} = \frac{\langle x_1^2 x_3\rangle}{\langle x_1^2\rangle\langle x_3\rangle} = \frac{\langle x_{10}^4\rangle}{\langle x_{10}^2\rangle^2}.$$

However x_{10} is Gaussian distributed and hence, $\langle x_{10}^4\rangle/\langle x_{10}^2\rangle^2 = 3$, a property of a Gaussian random variable with vanishing mean. Similarly one can show that,

$$\frac{E_{123}}{E_{12}E_3} = \frac{E_{113}}{E_{11}E_3} = 3. \tag{5.8}$$

It is quite apparent that the functional dependence (5.6) is not valid at small times. The manner in which these ratios evolve from 3 at small times to 1 at large times is the essence of the closure problem. Thus there is no simple closure, in which higher-order moments are represented as algebraic functions of lower-order ones, valid throughout the time evolution.

6. Conclusions

We have treated the evolution of Rayleigh–Bénard convection by considering the random thermodynamic fluctuations to be the driving mechanism for the onset of convection. By the development of a Monte Carlo simulation technique, we have been able to construct the statistics of the evolution from a finite population of realizations. We find three basic stages in the evolution process which reflect the time regions in which (i) forcing is important, (ii) forcing is unimportant and each realization grows exponentially (iii) nonlinearities lead to steady convection. These three regimes, determined by Monte Carlo calculations, have many features in common with results available for simple model equations such as (5.5). Furthermore we have shown that there is no algebraic closure to the moment equations which is uniformly valid in time. Thus, accurate theory for randomly-forced instability problems of more complexity, e.g. high-Rayleigh-number convection, three-dimensional convection, or instabilities of time-dependent states will require considerable computation.

We wish to point out two interesting features of randomly-forced instability problems which are absent in a deterministic description. The first relates to the evolution of a large population of convection cells which would be present, for example, in containers of large aspect ratio. Consider the evolutions shown in figure 2. After the initial period, each realization evolves in a deterministic manner, given by solutions to the so-called amplitude equations of nonlinear stability theory. However, since all realizations are not equally probable, and the probability density function also

† Equations (5.7) also contains another term proportional to $e^{\lambda_2 t}$, where λ_2 is a second eigenvalue of matrix a_{ij} in (4.1). However, $\lambda_2 < 0$ while $\lambda_1 > 0$, and $e^{\lambda_2 t}$ may be neglected in comparison during the intermediate stage.

evolves in time, the mean observable transport evolves in a manner *distinct from* the deterministic solution; refer to figure 2. Thus in experiments with containers of large aspect ratio, multiple experiments will give the same mean evolution, and this should differ from that predicted deterministically.

The second point relates to the onset time for observable convection. Again, with reference to figure 2, it is clear that since different realizations begin to convect at different times, the 'onset time' may be described in a statistical sense as follows. Since all realizations do not have equal probability of occurrence, one may define the probability density $p(t; Nu^*)$ that at a given time, t, the transport rate for a realization is equal to Nu^*. The most probable time of observable convection is therefore the mean over this distribution. We will apply these ideas in a later paper dealing with the onset time of convection in fluid layers heated in a time-dependent manner.

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Appendix. Eigenvalues and eigenvectors $\mathbf{A}_{cp}$

Setting $\det(\mathbf{A}_{cp} - \lambda_{cp}\mathbf{I}) = 0$ where $\mathbf{I}$ is the unit matrix, we get the characteristic equation

$$\left[\lambda_{cp}^2 + 2(c^2\alpha^2 + p^2\pi^2)\left(1+\frac{1}{Pr}\right)\lambda_{cp} + \frac{4(c^2\alpha^2+p^2\pi^2)^2}{Pr} + \frac{4Rc^2\alpha^2}{Pr}\left(1+\frac{1}{Pr}\right)\right] \times \left(\lambda_{cp} + (c^2\alpha^2+p^2\pi^2)\left(1+\frac{1}{Pr}\right)\right) = 0.$$

The three eigenvalues are,

$$\lambda_{cp}^{(1),(2)} = -(c^2\alpha^2+p^2\pi^2)\left(1+\frac{1}{Pr}\right) \pm \left[(c^2\alpha^2+p^2\pi^2)^2\left(1+\frac{1}{Pr}\right)^2 - \frac{4(c^2\alpha^2+p^2\pi^2)^2}{Pr} + \frac{4R}{Pr}\frac{c^2\alpha^2}{c^2\alpha^2+p^2\pi^2}\right]^{\frac{1}{2}},$$

$$\lambda_{cp}^{(3)} = -(c^2\alpha^2+p^2\pi^2)\left(1+\frac{1}{Pr}\right).$$

Then the matrix of eigenvectors has the form,

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ \dfrac{\lambda_{cp}^{(1)} - a_{11}}{a_{12}} & \dfrac{\lambda_{cp}^{(2)} - a_{11}}{a_{12}} & \dfrac{\lambda_{cp}^{(3)} - a_{11}}{a_{12}} \\ \dfrac{a_{32}}{a_{12}}\dfrac{(a_{11}-\lambda_{cp}^{(1)})}{(a_{33}-\lambda_{cp}^{(1)})} & \dfrac{a_{32}}{a_{12}}\dfrac{(a_{11}-\lambda_{cp}^{(2)})}{(a_{33}-\lambda_{cp}^{(2)})} & \dfrac{a_{32}}{a_{12}}\dfrac{(a_{11}-\lambda_{cp}^{(3)})}{(a_{33}-\lambda_{cp}^{(3)})} \end{bmatrix},$$

where a_{ij} is the ij component of matrix $\mathbf{A}_{cp}$.

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